

DUALITY AND HEREDITARY KÖNIG-EGERVÁRY SET-SYSTEMS

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ABSTRACT. A König-Egerváry graph is a graph G satisfying $\alpha(G) + \mu(G) = |V(G)|$, where $\alpha(G)$ is the cardinality of a maximum independent set and $\mu(G)$ is the matching number of G . Such graphs are those that admit a matching between $V(G) - \bigcup \Gamma$ and $\bigcap \Gamma$ where Γ is a set-system comprised of maximum independent sets satisfying $|\bigcup \Gamma'| + |\bigcap \Gamma'| = 2\alpha(G)$ for every set-system $\Gamma' \subseteq \Gamma$; in order to improve this characterization of a König-Egerváry graph, we characterize *hereditary König-Egerváry set-systems* (HKE set-systems, here after).

An *HKE* set-system is a set-system, F , such that for some positive integer, α , the equality $|\bigcup \Gamma| + |\bigcap \Gamma| = 2\alpha$ holds for every non-empty subset, Γ , of F .

We prove the following theorem: Let F be a set-system. F is an HKE set-system if and only if the equality $|\bigcap \Gamma_1 - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup \Gamma_1|$ holds for every two non-empty disjoint subsets, Γ_1, Γ_2 of F .

This theorem is applied in [2],[1].

1. INTRODUCTION

In this section we give the basic definitions and motivate the study of HKE set-systems.

For a uniform set-system, F , we denote by $\alpha(F)$ the cardinality of a set in F . We write α , when F is clear from the context.

The following definition contradicts the definition of a König-Egerváry set-system in [3].

Definition 1.1. Let F be a uniform set-system. F is said to be a *König-Egerváry set-system* (KE set-system in short), if the following equality holds:

$$|\bigcup F| + |\bigcap F| = 2\alpha(F).$$

Definition 1.2. An *HKE* set-system is a set-system, F , such that for some positive integer, α , the equality

$$|\bigcup \Gamma| + |\bigcap \Gamma| = 2\alpha$$

holds for every non-empty subset, Γ , of F .

Proposition 1.3. *Every HKE set-system is a uniform set-system. So a set-system F is HKE if and only if each subset Γ of F is KE.*

Proof. Let F be an HKE set-system and let $A \in F$. By Definition 1.2, where we substitute $\Gamma = \{A\}$, we have $|A| = \alpha$. So F is a uniform set-system and $\alpha = \alpha(F)$. ◻

Proposition 1.4. *Let F be a uniform set-system. If $|F| \leq 2$ then it is an HKE set-system.*

Proof. It is clear when $|F| = 1$. So assume $|F| = 2$, $F = \{A, B\}$. Take a non-empty sub-set-system Γ of F . Without loss of generality, $\Gamma = F$. So

$$|\bigcup \Gamma| + |\bigcap \Gamma| = |A \cup B| + |A \cap B| = |A| + |B| = 2\alpha(F).$$

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Theorem 1.5 and Propositions 1.6, 1.7 exemplifies the usefulness of HKE set-systems in the study of König-Egerváry graphs.

The following theorem is a restatement of [3, Theorem 2.6] in our notation.

Theorem 1.5. *G is a König-Egerváry graph if and only if there is a matching between $V(G) - \bigcup \Gamma$ and $\bigcap \Gamma$, where Γ is an HKE set-system comprised of maximum independent sets.*

Proposition 1.6. *Let G be a KE graph. Then $\Omega(G)$ is an HKE set-system.*

Proof. By [4, Theorem 3.6] and [4, Corollary 2.8].

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Proposition 1.7. *Every KE set-system that is comprised of maximum independent sets of some graph is an HKE set-system.*

Proof. By [4, Corollaries 2.7 and 2.9].

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2. HKE SET-SYSTEMS AND DUALITY

In this section, we characterize the HKE set-systems; consequently, we get a new characterization of a König-Egerváry graph. Proposition 2.2 is a weak version of Theorem 2.5, where we add the assumption, that the set-system is uniform.

In order to state Proposition 2.2, Theorem 2.5 and Corollary 2.6, we present the following equality:

Equality 2.1.

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup \Gamma_1|.$$

Proposition 2.2. *Let F be a uniform set-system.*

The following are equivalent:

- (1) *F is an HKE set-system.*
- (2) *Equality 2.1 holds for every two non-empty disjoint sub-set-systems, Γ_1, Γ_2 of F ,*
- (3) *Equality 2.1 holds for every two non-empty disjoint sub-set-systems, Γ_1, Γ_2 of F with $\Gamma_1 \cup \Gamma_2 = F$.*

The argument of Proposition 2.2 is based on the following exercise:

Exercise 2.3. Assume that $\{A, B, C, D\}$ is an HKE set-system (so in particular $\{A, B, C\}$ is an HKE set-system). Prove:

- (1) $|A - B - C| = |B \cap C - A|$. A clue: $A - B - C = (A \cup B \cup C) - (B \cup C)$ and $B \cap C - A = (B \cap C) - (A \cap B \cap C)$.
- (2) $|A \cap B - C - D| = |C \cap D - A - B|$. A clue: $A \cap B - C - D = (A - C - D) - (A - B - C - D)$. Apply Clause (1).

We now prove Proposition 2.2.

Proof. (1) \Rightarrow (2) : We prove it by induction on $r = |\Gamma_1|$.

Case a: $r = 1$, so $\Gamma_1 = \{A^*\}$ for some set A^* . In this case, we apply the idea of Exercise 2.3(1).

We should prove that

$$|A^* - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - A^*|,$$

namely,

$$|\bigcup \Gamma_2 \cup A^*| - |\bigcup \Gamma_2| = |\bigcap \Gamma_2| - |\bigcap \Gamma_2 \cap A^*|,$$

or equivalently,

$$|\bigcup \Gamma_2 \cup A^*| + |\bigcap \Gamma_2 \cap A^*| = |\bigcap \Gamma_2| + |\bigcup \Gamma_2|.$$

But by Clause (1), each side of this equality equals 2α .

Case a: $r > 1$. In this case, we apply the idea of Exercise 2.3(2). We fix $A^* \in \Gamma_1$. First we write three trivial equalities, for convenience:

$$\bigcap (\Gamma_1 - \{A^*\}) = \{x : x \in A \text{ for every } A \in \Gamma_1 \text{ with } A \neq A^*\},$$

$$\bigcup (\Gamma_1 - \{A^*\}) = \{x : x \in A \text{ for some } A \in \Gamma_1 \text{ with } A \neq A^*\}$$

and

$$\bigcap (\Gamma_1 \cup \{A^*\}) = A^* \cap \bigcap \Gamma_1.$$

We now begin the computation.

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2| = |\bigcap (\Gamma_1 - \{A^*\}) - \bigcup \Gamma_2| - |\bigcap (\Gamma_1 - \{A^*\}) - \bigcup (\Gamma_2 \cup \{A^*\})|.$$

The right side of this equality is a subtraction of two summands. Since $|\Gamma_1 - \{A^*\}| < |\Gamma_1|$, we may apply the induction hypothesis on each summand:

$$|\bigcap (\Gamma_1 - \{A^*\}) - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup (\Gamma_1 - \{A^*\})|$$

and

$$|\bigcap (\Gamma_1 - \{A^*\}) - \bigcup (\Gamma_2 \cup \{A^*\})| = |\bigcap (\Gamma_2 \cup \{A^*\}) - \bigcup (\Gamma_1 - \{A^*\})|.$$

By the three last equalities we get:

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup (\Gamma_1 - \{A^*\})| - |\bigcap (\Gamma_2 \cup \{A^*\}) - \bigcup (\Gamma_1 - \{A^*\})|.$$

So

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup \Gamma_1|.$$

Equality 2.1 is proved, so Clause (2) is proved.

(2) \Rightarrow (1) : Let Γ be a non-empty subset of F . Fix $D \in \Gamma$. Since F is a uniform set-system, $|D| = \alpha$ (this is the unique place where we use the assumption that F is a uniform set-system, but we eliminate this assumption later). Therefore it is enough to prove that

$$|\bigcup \Gamma| + |\bigcap \Gamma| = 2|D|,$$

or equivalently,

$$|\bigcup \Gamma - D| = |D - \bigcap \Gamma|.$$

Let H be the set of ordered pairs $\langle \Gamma_1, \Gamma_2 \rangle$ of non-empty disjoint subsets of Γ such that $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $D \in \Gamma_2$.

By Clause (2),

$$\sum_{\langle \Gamma_1, \Gamma_2 \rangle \in H} |\bigcap \Gamma_1 - \bigcup \Gamma_2| = \sum_{\langle \Gamma_1, \Gamma_2 \rangle \in H} |\bigcap \Gamma_2 - \bigcup \Gamma_1|.$$

So it is enough to prove the following two equalities:

$$|\bigcup \Gamma - D| = \sum_{\langle \Gamma_1, \Gamma_2 \rangle \in H} |\bigcap \Gamma_1 - \bigcup \Gamma_2|$$

and

$$|D - \bigcap \Gamma| = \sum_{\langle \Gamma_1, \Gamma_2 \rangle \in H} |\bigcap \Gamma_2 - \bigcup \Gamma_1|.$$

Since their proofs are dual, we prove the first equality only.

$$\bigcup \Gamma - D = \bigcup_{\langle \Gamma_1, \Gamma_2 \rangle \in H} (\bigcap \Gamma_1 - \bigcup \Gamma_2),$$

(on the one hand, if $x \in \bigcup \Gamma - D$ then for $\Gamma_1 = \{A \in \Gamma : x \in A\}$ and $\Gamma_2 = \{A \in \Gamma : x \notin A\}$ we have $x \in \bigcap \Gamma_1 - \bigcup \Gamma_2$ and $\langle \Gamma_1, \Gamma_2 \rangle \in H$. On the other hand, assume that $x \in \bigcap \Gamma_1 - \bigcup \Gamma_2$ for some $\langle \Gamma_1, \Gamma_2 \rangle \in H$. Then $x \in \bigcup \Gamma$ (because $x \in \bigcap \Gamma_1$ and $\emptyset \neq \Gamma_1 \subseteq \Gamma$) and $x \notin D$ (because $x \notin \bigcup \Gamma_2$ and $D \subseteq \bigcup \Gamma_2$). So $x \in \bigcup \Gamma - D$). Therefore

$$|\bigcup \Gamma - D| = \sum_{\langle \Gamma_1, \Gamma_2 \rangle \in H} |\bigcap \Gamma_1 - \bigcup \Gamma_2|,$$

because this is a sum of cardinalities of disjoint sets (if $\langle \Gamma_1, \Gamma_2 \rangle$ and $\langle \Gamma_3, \Gamma_4 \rangle$ are two different pairs in H then there is no element $x \in (\bigcap \Gamma_1 - \bigcup \Gamma_2) \cap (\bigcap \Gamma_3 - \bigcup \Gamma_4)$. Otherwise, take $A \in \Gamma_1 - \Gamma_3$ (or vice versa). So $A \in \Gamma_4$. Hence, $x \in \bigcap \Gamma_1 \subseteq A$ and $x \notin \bigcup \Gamma_4 \supseteq A$, a contradiction).

The implication (2) \Rightarrow (1) is proved.

Since Clause (3) is a private case of Clause (2), it remains to prove (3) \Rightarrow (2). Let Γ_1, Γ_2 be two non-empty disjoint subsets of F . We should prove Equality 2.1 for these Γ_1 and Γ_2 , without assuming $\Gamma_1 \cup \Gamma_2 = F$. Let H be the set of disjoint pairs $\langle \Gamma_1^+, \Gamma_2^+ \rangle$ of F such that $\Gamma_1 \subseteq \Gamma_1^+$, $\Gamma_2 \subseteq \Gamma_2^+$ and $\Gamma_1^+ \cup \Gamma_2^+ = F$.

By Clause (3),

$$\sum_{\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H} |\bigcap \Gamma_1^+ - \bigcup \Gamma_2^+| = \sum_{\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H} |\bigcap \Gamma_2^+ - \bigcup \Gamma_1^+|.$$

So it remains to prove the following two equalities:

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2| = \sum_{\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H} |\bigcap \Gamma_1^+ - \bigcup \Gamma_2^+|$$

and

$$|\bigcap \Gamma_2 - \bigcup \Gamma_1| = \sum_{\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H} |\bigcap \Gamma_2^+ - \bigcup \Gamma_1^+|,$$

Since their proofs are dual, we prove the first equality only.

$$\bigcap \Gamma_1 - \bigcup \Gamma_2 = \bigcup_{\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H} (\bigcap \Gamma_1^+ - \bigcup \Gamma_2^+)$$

(On the one hand, if $x \in \bigcap \Gamma_1 - \bigcup \Gamma_2$ then for $\Gamma_1 = \{A \in \Gamma : x \in A\}$ and $\Gamma_2 = \{A \in \Gamma : x \notin A\}$, we have $x \in \bigcap \Gamma_1^+ - \bigcup \Gamma_2^+$ and the pair $\langle \Gamma_1^+, \Gamma_2^+ \rangle$ belongs to H . On the other hand, if $x \in \bigcap \Gamma_1^+ - \bigcup \Gamma_2^+$ for some $\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H$ then $x \in \bigcap \Gamma_1^+ \subseteq \bigcap \Gamma_1$ and $x \notin \bigcup \Gamma_2^+ \supseteq \bigcup \Gamma_2$. Hence, $x \in \bigcap \Gamma_1 - \bigcup \Gamma_2$). Therefore

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2| = \sum_{\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H} |\bigcap \Gamma_1^+ - \bigcup \Gamma_2^+|,$$

because it is a sum of disjoint sets. \dashv

The following proposition eliminates the assumption that F is a uniform set-system.

Proposition 2.4. *Clause (3) of Proposition 2.2 implies that F is a uniform set-system.*

Proof. Define

$$\alpha = \frac{|\bigcup F| + |\bigcap F|}{2}.$$

Let $D \in F$. We prove that $|D| = \alpha$. Let P denote the family of partitions $\{\Gamma_1, \Gamma_2\}$ of F into two non-empty subsets.

Every element in $\bigcup F$ is in $\bigcap \Gamma_1 - \bigcup \Gamma_2$ for some partition $\{\Gamma_1, \Gamma_2\} \in P$ or in $\bigcap F$.

Let

$$P_1 = \{\{\Gamma_1, \Gamma_2\} \in P : D \in \Gamma_1\}$$

and

$$P_2 = \{\{\Gamma_1, \Gamma_2\} \in P : D \notin \Gamma_1\}.$$

Define

$$x = \sum_{\{\Gamma_1, \Gamma_2\} \in P_1} |\bigcap \Gamma_1 - \bigcup \Gamma_2|$$

and

$$y = \sum_{\{\Gamma_1, \Gamma_2\} \in P_2} |\bigcap \Gamma_1 - \bigcup \Gamma_2|.$$

By Clause (3) of Proposition 2.2, we have $x = y$.

It is easy to check the following three equalities:

- (1) $|\bigcup F| = x + y + |\bigcap F| = 2x + |\bigcap F|$,
- (2) $|D| = x + |\bigcap F|$ and
- (3) $|\bigcup F| + |\bigcap F| = 2\alpha$ (by the definition of α).

By Equalities (1)-(3), $|D| = \alpha$. Since D is an arbitrary set in F , F is a uniform set-system. \dashv

Theorem 2.5. *Let F be a set-system.*

The following are equivalent:

- (1) *F is an HKE set-system.*
- (2) *Equality 2.1 holds for every two non-empty disjoint sub-set-systems, Γ_1, Γ_2 of F ,*
- (3) *Equality 2.1 holds for every two non-empty disjoint sub-set-systems, Γ_1, Γ_2 of F with $\Gamma_1 \cup \Gamma_2 = F$.*

Proof. By Proposition 2.2, it is enough to prove that each clause implies that F is a uniform set-system. By Proposition 1.3, Clause (1) implies that F is a uniform set-system. By Proposition 2.4 Clause (3) implies that F is a uniform set-system. But Clause (2) implies Clause (3). \dashv

Corollary 2.6. *Let G be a graph. The following are equivalent:*

- (1) G is a KE graph.
- (2) For some non-empty HKE set-system $F \subseteq \Omega(G)$, there is a matching $M : V[G] - \bigcup F \rightarrow \bigcap F$ and Equality 2.1 holds for every two non-empty disjoint sub-set-systems, Γ_1, Γ_2 of F .
- (3) For some non-empty HKE set-system $F \subseteq \Omega(G)$, there is a matching $M : V[G] - \bigcup F \rightarrow \bigcap F$ and Equality 2.1 holds for every two non-empty disjoint sub-set-systems, Γ_1, Γ_2 of F with $\Gamma_1 \cup \Gamma_2 = F$.

Proof. By Theorem 2.5 and Theorem 1.5. \dashv

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